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THE NON-LINEAR CRACK THEORY†

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Results obtained in recent years in an attempt to construct a general (physically and geometrically) non-linear crack theory are described. © 1999 Elsevier Science Ltd. All rights reserved.

Underlying the widely used linear crack theory there is a clear lack of correspondence between the object being investigated and the mathematical apparatus used. On the one hand, the neighbourhoods of singular points of the region (the tips of the cracks and linear inclusions, corner points of cuts and inclusions, points where point forces and moments are applied, and the axes of dislocations and disclinations), in the neighbourhood of which large deformations and rotations occur, are of the greatest interest. On the other hand, the linear theory of elasticity (or the geometrically linear theory of plasticity) is used, for which this situation is obviously "counterindicated".

It would seem that the linear theory is inapplicable here. However, long and not entirely unsuccessful experience in using it cautions against such a categoric conclusion. It is therefore of interest to investigate the area in which the linear theory can be justifiably employed.

When speaking of the non-linear crack theory we usually have in mind a physical non-linearity (most often plasticity). Geometrical non-linearity is considered in a comparatively small number of papers. Often different simplifying assumptions are introduced, the correctness and area of applicability of which are not made clear. On the whole, existing publications are of a fragmentary nature. Also, the actual stresses used here are unsuitable when considering singular problems [1–3].

An attempt has been made to construct a consistent general (physically and geometrically) non-linear crack theory [1, 2]. Particular attention has been given to the least-investigated geometrical non-linearity. A comparison of the exact solutions of standard boundary-value problems, obtained using the linear and non-linear theory of elasticity, was used as the basis.

Attempts to find exact solutions required a large amount of preliminary work to set up the simplest version of the non-linear theory of elasticity (but without loss of generality). First this was a further development of the complex method, the search for new types of boundary conditions, justification of the choice of the energy pair of tensors, and new versions of two-dimensional problems. The latter include the unified plane problem, generalized antiplane deformation, axisymmetric deformation of solids of revolution, the Kirchhoff and Reissner theory of shells, and Volterra dislocations.

The introduction of new models of materials (laws of elasticity) however, was decisive for obtaining accurate solutions of non-linear boundary-value problems. These included a reduced standard model of the neo-Hookeian, low-compression and hybrid type (which satisfy the macrolaw of compressibility of a material and the microlaw of particle interaction). The results obtained using these models were confirmed by the asymptotic forms in the neighbourhoods of singular points, obtained using complex invariant integrals, which were also extended to rigid inclusions.

This paper gives a brief description of the results we have obtained, including new results not published previously. The notation and terminology employed are the same as that used in [4].

1. In the non-linear theory of elasticity the following are widely employed: the complex coordinates

$$\zeta = \mathring{x}_1 + i\mathring{x}_2, \quad \bar{\zeta} = \mathring{x}_1 - i\mathring{x}_2, \quad z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2$$

differentiation with respect to these

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial \mathring{x}_1} - i\frac{\partial}{\partial \mathring{x}_2}\right), \quad \frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial \mathring{x}_1} + i\frac{\partial}{\partial \mathring{x}_2}\right)$$

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and the complex components of the tensors

$$T_1 = t_{11} + t_{22} + i(t_{12} - t_{21}), \quad T_2 = t_{11} - t_{22} + i(t_{12} + t_{21})$$
$$T_3 = t_{13} + it_{23}, \quad T_4 = t_{31} + it_{32}, \quad T_5 = t_{33}$$

Here and below the degree sign is given to quantities which refer to the undeformed configuration of a body, while quantities without this sign refer to the deformed configuration. Thus, $\mathring{x}_i$ and x_i are rectangular Cartesian coordinates of a point mass before and after deformation.

2. The following equations apply to generalized plane deformation [1, 4]

$$z = z(\zeta, \bar{\zeta}), \quad x_3 = \lambda \mathring{x}_3 \tag{2.1}$$

($\lambda = \lambda_3 =$ const is the multiplicity of the elongations in a transverse direction), and the homogeneous equilibrium equation

$$\frac{\partial\{F^{-1}J\Sigma\}_1}{\partial\bar{\zeta}} + \frac{\partial\{F^{-1}J\Sigma\}_2}{\partial\zeta} = 0 \tag{2.2}$$

Here $\{F^{-1} \cdot J\Sigma\}_i$ are the complex components of the nominal-stress tensor.
We also have the static boundary condition

$$\{F^{-1}J\Sigma\}_1 e^{i\mathring{\gamma}} + \{F^{-1} \cdot J\Sigma\}_2 e^{i\mathring{\gamma}} = 2[\sigma_{\mathring{v}\mathring{v}}(\mathring{s}) + i\sigma_{\mathring{v}\mathring{t}}(\mathring{s})]e^{i\mathring{\gamma}} \tag{2.3}$$

and the rigid-edge condition†

$$\frac{\partial z}{\partial\zeta} e^{i\mathring{\gamma}} - \frac{\partial z}{\partial\bar{\zeta}} e^{-i\mathring{\gamma}} = e^{i\mathring{\gamma}} \tag{2.4}$$

Here $\mathring{\gamma}$ is the angle between the normal to the undeformed boundary contour of the region (Fig. 1), and $\sigma_{\mathring{v}\mathring{v}}(\mathring{s})$ and $\sigma_{\mathring{v}\mathring{v}}(\mathring{s})$ are the normal and shear stresses, specified on the contour.

The tensor of the conventional values $\mathring{\Sigma}$ (the symmetrical Biot tensor) and the tensor of the actual stresses Σ (the Cauchy stress tensor) are related to the tensor of the nominal stresses $\{F^{-1} \cdot J\Sigma\}$ by the equations

$$\begin{aligned}
\mathring{\Sigma}_1 &\equiv \mathring{\sigma}_{11} + \mathring{\sigma}_{22} = \operatorname{Re}\left[\left|\frac{\partial z}{\partial\zeta}\right|^{-1} \overline{\frac{\partial z}{\partial\zeta}} \{F^{-1}J\Sigma\}_1\right] \\
\mathring{\Sigma}_2 &\equiv \mathring{\sigma}_{11} - \mathring{\sigma}_{22} + i2\sigma_{12} = \left|\frac{\partial z}{\partial\zeta}\right|^{-1} \overline{\frac{\partial z}{\partial\zeta}} \{F^{-1}J\Sigma\}_2 \\
\mathring{\Sigma}_5 &\equiv \mathring{\sigma}_{33} = \{F^{-1}J\Sigma\}_{33}
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\lambda\Delta\Sigma_1 &\equiv \lambda\Delta(\sigma_{11} + \sigma_{22}) = \overline{\frac{\partial z}{\partial\zeta}} \{F^{-1} \cdot J\Sigma\}_1 + \overline{\frac{\partial z}{\partial\zeta}} \{F^{-1}J\Sigma\}_2 \\
\lambda\Delta\Sigma_2 &\equiv \lambda\Delta(\sigma_{11} - \sigma_{22} + i2\sigma_{12}) = \frac{\partial z}{\partial\bar{\zeta}} \{F^{-1} \cdot J\Sigma\}_1 + \frac{\partial z}{\partial\zeta} \{F^{-1}J\Sigma\}_2 \\
\lambda\Delta\Sigma_5 &\equiv \lambda\Delta\sigma_{33} = \lambda\{F^{-1} \cdot J\Sigma\}_{33}
\end{aligned} \tag{2.6}$$

Here

$$\Delta = \frac{\partial z}{\partial\zeta} \overline{\frac{\partial z}{\partial\zeta}} - \frac{\partial z}{\partial\bar{\zeta}} \overline{\frac{\partial z}{\partial\bar{\zeta}}} = \left|\frac{\partial z}{\partial\zeta}\right|^2 - \left|\frac{\partial z}{\partial\bar{\zeta}}\right|^2 \tag{2.7}$$

†It implies the continuity of displacements at the interface between a deformable body and inclusion (Editor's note).

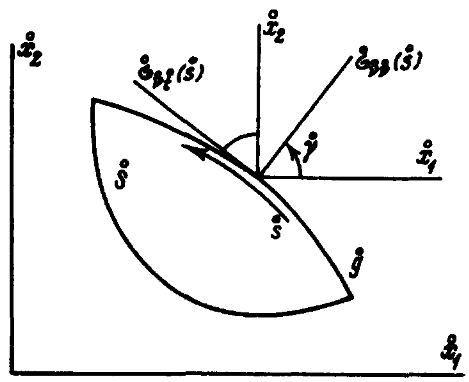

Fig. 1.

is the multiplicity of the variation of the area of the region, which is related to the multiplicity of the volume J by the equation

$$J = \lambda\Delta \tag{2.8}$$

For an incompressible isotropic material we have the relations

$$\{F^{-1}\cdot J\Sigma\}_1 = \left[\left|\frac{\partial z}{\partial \zeta}\right|^{-1}\frac{\partial \Phi}{\partial|\partial z/\partial\zeta|}+q\right]\frac{\partial z}{\partial \zeta}$$

$$\{F^{-1}\cdot J\Sigma\}_2 = \left[\left|\frac{\partial z}{\partial \bar\zeta}\right|^{-1}\frac{\partial \Phi}{\partial|\partial z/\partial\bar\zeta|}-q\right]\frac{\partial z}{\partial \bar\zeta}$$

$$\{F^{-1}\cdot J\Sigma\}_{33} = \frac{\partial\Phi}{\partial\lambda}+\frac{2q}{\lambda^2} \tag{2.9}$$

$$J = 1$$

$$\Phi = \Phi\left(\left|\frac{\partial z}{\partial\zeta}\right|, \left|\frac{\partial z}{\partial\bar\zeta}\right|, \lambda\right)$$

Here Φ is the elastic potential (the deformation energy density), while $q(\mathring{x}_1, \mathring{x}_2)$ is a statically defined function, of the uniform compression type. For a compressible material we must omit q and the incompressibility condition in (2.9). The relations for the plane stressed state of a thin plate differ only slightly from the relations given above [4].

3. The following equation applies to generalized antiplane deformation

$$z = c\zeta, \quad x_3 = \lambda\mathring{x}_3 + w(\zeta, \bar\zeta) \tag{3.1}$$

Here c and λ are positive constants and $w(\zeta, \bar\zeta)$ is a real function.

Unlike the linear case, for which there are only two components of the stress (σ_{13}, σ_{23}), all six components are present here. Moreover, in the general case, the antiplane problem is overdefined. In the case of an incompressible material this overdefinition disappears for a "neo-Hookeian" material

$$\Phi = \Phi(I_c) = \Phi\left(\lambda^2 + 2c^2 + 4\frac{\partial w}{\partial\zeta}\overline{\frac{\partial w}{\partial\zeta}}\right) \tag{3.2}$$

Here I_c is the first invariant of the Cauchy strain tensor. A special case of this law is the neo-Hookeian material for which

$$\Phi = \frac{E}{6}(I_c - 3) = \frac{E}{6}\left(\lambda^2 + 2c^2 + 4\frac{\partial w}{\partial \zeta}\frac{\overline{\partial w}}{\partial \zeta}\right) \quad (3.3)$$

The following expressions for the required quantities in terms of the function of a complex variable $\Psi(\zeta)$ correspond to the latter case

$$w = \frac{1}{2}\left[\int \Psi(\zeta)d\zeta + \overline{\int \Psi(\zeta)d\zeta}\right]$$

$$\Sigma_3 \equiv \sigma_{13} + i\sigma_{23} = \frac{1}{3}E\lambda^{-\frac{1}{2}}\overline{\Psi(\zeta)}, \quad \sigma_{33} = \frac{1}{3}E[\Psi(\zeta)\overline{\Psi(\zeta)} + \lambda^2 - \lambda^{-1}]$$

$$\sigma_{11} = \sigma_{12} = \sigma_{22} = 0$$

$$\mathring{\Sigma}_1 \equiv \mathring{\sigma}_{11} + \mathring{\sigma}_{22} = \frac{E}{3}\left[\frac{\cos\omega}{\lambda + \lambda^{-\frac{1}{2}}}\Psi(\zeta)\overline{\Psi(\zeta)}\right]$$

$$\mathring{\Sigma}_2 \equiv \mathring{\sigma}_{11} - \mathring{\sigma}_{22} + i2\mathring{\sigma}_{12} = \frac{E}{3}\frac{\cos\omega}{\lambda + \lambda^{-\frac{1}{2}}}\overline{\Psi(z)}^2 \quad (3.4)$$

$$\mathring{\Sigma}_3 \equiv \mathring{\sigma}_{13} + i\mathring{\sigma}_{23} = \frac{E}{3}\cos\omega\overline{\Psi(\zeta)}$$

$$\mathring{\sigma}_{33} = \frac{E}{3}\cos\omega\left[-\frac{1}{\lambda^{\frac{1}{2}} + \lambda}\Psi(\zeta), \ \overline{\Psi(\zeta)} + \lambda - \lambda^{-2}\right]$$

$$\left(\cos\omega = \frac{\lambda + \lambda^{-\frac{1}{2}}}{\sqrt{(\lambda + \lambda^{-\frac{1}{2}})^2 + 4\Psi(\zeta)\overline{\Psi(\zeta)}}}\right)$$

The homogeneous static condition has the form

$$\Psi e^{i\mathring{\gamma}} + \overline{\Psi e^{i\mathring{\gamma}}} = 0 \quad (3.5)$$

while the rigid-edge condition is

$$\Psi e^{i\mathring{\gamma}} - \Psi e^{-i\mathring{\gamma}} = 0, \quad \lambda = 1 \quad (3.6)$$

It is essential that, for the general non-linearity of the plane and anti-plane problems, the resolvents (2.3), (2.4), (3.5) and (3.6) are linear, so that the whole arsenal of the methods of linear theory are suitable for solving the boundary-value problems. An anisotropic material is considered in [5, 6, 7].

4. In the plane problem a material with an elastic potential

$$\Phi = \sigma^*\left|\frac{\partial z}{\partial \zeta}\right|^2 + \alpha\left|\frac{\partial z}{\partial \bar{\zeta}}\right|^2 \quad (4.1)$$

was considered in detail. Here σ^* and α are the constants of elasticity. Along the principal strain axes the following relation exists for this material between the principal extension multiplicities λ_i and the principal conventional stresses $\mathring{\sigma}_i$:

$$\mathring{\sigma}_1 - \sigma^* = \sigma^*\frac{(\lambda_1 - 1) + (\lambda_2 - 1)}{2} + \alpha\frac{(\alpha_1 - 1) - (\lambda_2 - 1)}{2}$$

$$\mathring{\sigma}_2-\sigma^* = \sigma^*\frac{(\lambda_1-1)+(\lambda_2-1)}{2}-\alpha\frac{(\alpha_1-1)-(\lambda_2-1)}{2}$$

when $\lambda_1 = \lambda_2 = 1$, i.e. when there is no strain, $\mathring{\sigma}_1 = \mathring{\sigma}_2 = \sigma^*$. Hence, we are dealing with a prestressed linear material. It can be regarded as a reduced standard material (it is a harmonic, semi-linear John material) with

$$\sigma^* = \frac{E}{(1+\nu)(1-2\nu)}, \quad \alpha = \frac{E}{1+\nu}$$

By taking potential (4.1) we can introduce the Goursat–Kolosov functions $\Phi(\zeta)$, $\Psi(\zeta)$, in terms of which the required functions are expressed as

$$z = \int\Phi(\zeta)d\zeta + \overline{\int\Psi(\zeta)d\zeta}, \quad \frac{\partial z}{\partial\zeta} = \Phi(\zeta), \quad \frac{\partial z}{\partial\bar{\zeta}} = \overline{\Psi(\zeta)}$$

$$\{F^{-1}\cdot J\Sigma\}_1 = 2\sigma^*\Phi(\zeta), \quad \{F^{-1}\cdot J\Sigma\}_2 = 2\alpha\overline{\Psi(\zeta)} \tag{4.2}$$

$$\mathring{\Sigma}_1 = 2\sigma^*|\Phi(\zeta)|, \quad \mathring{\Sigma}_2 = 2\alpha\overline{\Phi(\zeta)\Psi(\zeta)}|\Phi(\zeta)|^{-1}$$

$$\lambda\Delta\Sigma_1 = 2\left[\sigma^*\Phi(\zeta)\overline{\Phi(\zeta)} + \alpha\Psi(\zeta)\overline{\Psi(\zeta)}\right]$$

$$\lambda\Delta\Sigma_2 = 2(\sigma^*+\alpha)\Phi(\zeta)\overline{\Psi(\zeta)}$$

$$\left(\Delta = |\Phi(\zeta)|^2 - |\Psi(\zeta)|^2\right)$$

The static boundary condition takes the form

$$\sigma^*\Phi(\zeta)e^{i\vartheta} + \alpha\overline{\Psi(\zeta)e^{i\vartheta}} = [\sigma_{\vartheta\vartheta}(\mathring{s}) + i\sigma_{\vartheta s}(\mathring{s})]e^{i\vartheta} \tag{4.3}$$

while the rigid-edge condition takes the form

$$\Phi(\zeta)e^{i\mathring{\gamma}} - \overline{\Psi(\zeta)e^{i\mathring{\gamma}}} = e^{i\mathring{\gamma}} \tag{4.4}$$

The linear resolvents (4.3) and (4.4) for the general non-linearity of the problem enable us to obtain exact solutions of standard boundary-value problems. By comparing them with the corresponding solutions obtained using the linear theory we can clarify the effect of geometrical non-linearity "in pure form". This also applies to generalized antiplane strain for a neo-Hookeian material.

5. The relations obtained in Sections 2–4 were used primarily to investigate the effect of geometrical non-linearity and the unsuitability of the actual stresses for use in singular problems.

Thus, in the plane problem we have the following relations for the actual stresses in the neighbourhood of the tip of a rectangular crack in a compressible material (Fig. 2).

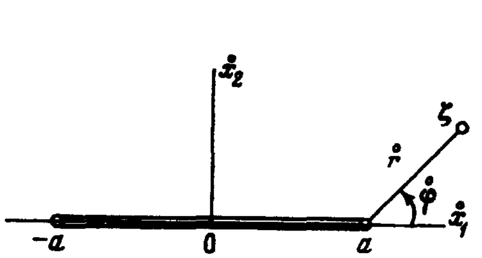

Fig. 2.

For a tensile crack

$$\frac{\sigma_{22}}{\sigma^*} \sim -\frac{1-2\nu}{\nu}\lambda^{-1}, \quad \frac{\sigma_{11}}{\sigma^*} \sim \frac{\sigma_{12}}{\sigma^*} \sim 0 \tag{5.1}$$

For a transverse-shear crack

$$\frac{\sigma_{11}}{\sigma^*} \sim -\frac{1-2\nu}{\nu}\lambda^{-1}, \quad \frac{\sigma_{22}}{\sigma^*} \sim \frac{\sigma_{12}}{\sigma^*} \sim 0 \tag{5.2}$$

In the case of an incompressible material we have the following relations.
For a tensile crack

$$\frac{\sigma_{22}}{\sigma^*} \sim \frac{1-\nu}{1-2\nu}\left(\frac{\mathring{\sigma}_{22}^{\infty}}{\sigma^*}\right)^2\left(\frac{a}{2\mathring{r}}\right)^1, \quad \frac{\sigma_{11}}{\sigma^*} \sim \frac{\sigma_{12}}{\sigma^*} \sim 0 \tag{5.3}$$

For a transverse-shear crack

$$\frac{\sigma_{11}}{\sigma^*} \sim \frac{1-\nu}{1-2\nu}\left(\frac{\mathring{\sigma}_{12}^{\infty}}{\sigma^*}\right)^2\left(\frac{a}{2\mathring{r}}\right)^1, \quad \frac{\sigma_{22}}{\sigma^*} \sim \frac{\sigma_{12}}{\sigma^*} \sim 0 \tag{5.4}$$

For a rigid linear inclusion in a compressible material (Fig. 2)

$$\frac{\Sigma_1}{\sigma^*} \equiv \frac{\sigma_{11}+\sigma_{22}}{\sigma^*} \sim \frac{4(1-\nu)|A|^2}{\lambda\left[\Delta/(a/(2\mathring{r}))^{1/2}\right]}\left(\frac{a}{2\mathring{r}}\right)^{1/2}$$

$$\frac{\Sigma_2}{\sigma^*} \equiv \frac{\sigma_{11}-\sigma_{22}+i2\sigma_{12}}{\sigma^*} \sim \frac{4(1-\nu)A^2}{\lambda\left[\Delta/(a/(2\mathring{r}))^{1/2}\right]}\left(\frac{a}{2\mathring{r}}\right)^{1/2}$$

$$[\Delta/(a/2\mathring{r})^{1/2}] = (A\bar{B}+\bar{A}B-\bar{A})e^{-i\mathring{\varphi}/2} + (\bar{A}B+A\bar{B}-A)e^{i\mathring{\varphi}/2}$$

$$\left\| \begin{matrix} A \\ B \end{matrix} \right\| = \mp\frac{1}{2} + \frac{\mathring{\sigma}_{11}^{\infty}+\mathring{\sigma}_{22}^{\infty}}{4\sigma^*} \pm \frac{\mathring{\sigma}_{11}^{\infty}-\mathring{\sigma}_{22}^{\infty}+i2\sigma_{12}^{\infty}}{4\alpha} \tag{5.5}$$

In the same way, in an incompressible material

$$\frac{\Sigma_1}{\sigma^*} \sim 4(1-\nu)|A|^2\left(\frac{a}{2\mathring{r}}\right)^1, \quad \frac{\Sigma_2}{\sigma^*} \sim 4(1-\nu)A^2\left(\frac{a}{2\mathring{r}}\right)^1 \tag{5.6}$$

For a longitudinal-shear crack in an incompressible (neo-Hookeian material)

$$\frac{\sigma_{13}}{\sigma_{23}^{\infty}} \sim -\sin\frac{\mathring{\varphi}}{2}\left(\frac{a}{2\mathring{r}}\right)^{1/2}, \quad \frac{\sigma_{23}}{\sigma_{23}^{\infty}} \sim \cos\frac{\mathring{\varphi}}{2}\left(\frac{a}{2\mathring{r}}\right)^{1/2}, \quad \frac{\sigma_{33}}{\sigma_{23}^{\infty}} \sim \lambda\frac{\sigma_{23}^{\infty}}{\mu}\left(\frac{a}{2\mathring{r}}\right)^1 \tag{5.7}$$

For a rigid linear inclusion in the case of antiplane strain

$$\frac{\sigma_{13}}{\sigma_{13}^{\infty}} \sim \cos\frac{\mathring{\varphi}}{2}\left(\frac{a}{2\mathring{r}}\right)^{1/2}, \quad \frac{\sigma_{23}}{\sigma_{13}^{\infty}} \sim \sin\frac{\mathring{\varphi}}{2}, \quad \frac{\sigma_{33}}{\sigma_{13}^{\infty}} \sim \frac{\sigma_{13}^{\infty}}{\mu}\left(\frac{a}{2\mathring{r}}\right)^{1/2} \tag{5.8}$$

The expressions obtained show the unsuitability of the actual stresses for use in the non-linear crack theory. From (5.1) and (5.2) it can be seen that for tensile and transverse-shear cracks in a compressible material the asymptotic form of the actual stresses is independent of the external load (σ_{22}^{∞}, $\overset{\circ}{\sigma}{}_{12}^{\infty}$). In addition, σ_{11} and σ_{22} become infinite for the traditionally "safe" value $\nu = 0$. Also, the minus sign does not correspond to the nature of the stress–strain state.

Moreover, expressions (5.1) and (5.2) also hold for a plane with lune and wedge-shaped cuts (inclusions). Hence the asymptotic form of the actual stresses also does not depend on the aperture angle of the cut (the inclusion).

One of the decisive arguments in favour of using the actual stresses is often the fact that they are finite at the tip of the crack. First of all, this is not so for physical non-linearity of the material. But also, for a linear material expressions (5.1)–(5.8) show that the singularity factor m (the exponent in the expression $(a/2\overset{\circ}{r})^m$) can take different values: $m = 0$ (finiteness of the stresses), $m = 1/2$ (a standard singularity, corresponding to the linear theory) and $m = 1$ (a singularity which considerably exceeds the standard one).

The list of above-mentioned defects can be enlarged considerably by extending it to dislocations and disclinations in crystals, point forces, and anisotropic and reinforced materials. As numerous examples show, the conventional stresses (the symmetrical Biot stresses) are free from these drawbacks. Defects of the actual stresses of a common character have been discussed in [2, 3]. It is concluded in [3] that "... Only fear of introducing greater disorder into already complicated terminology prevents the energy pair 'conventional stresses—elongation multiplicity' from being uniquely called the actual stresses and strains".

6. We will compare the behaviour of the actual stresses ($\overset{\circ}{\sigma}_{ij}$) and the linear stresses (σ_{ij}). Thus, in the case of a compressible material for a tensile crack we have, in local polar coordinates (Fig. 2), the asymptotic forms.

For conventional stresses

$$\left\| \begin{matrix} \overset{\circ}{\sigma}_{\overset{\circ}{r}\overset{\circ}{r}} \\ \overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}} \\ \overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{r}} \end{matrix} \right\| \sim \frac{\overset{\circ}{k}_I}{\sqrt{2\pi \overset{\circ}{r}}} \left\| \begin{matrix} \sin^2 \frac{\overset{\circ}{\varphi}}{2} \\ \cos^2 \frac{\overset{\circ}{\varphi}}{2} \\ \sin\frac{\overset{\circ}{\varphi}}{2}\cos\frac{\overset{\circ}{\varphi}}{2} \end{matrix} \right\| \quad (\overset{\circ}{k}_I = \overset{\circ}{\sigma}{}_{22}^{\infty}\sqrt{\pi a}) \tag{6.1}$$

For linear stresses

$$\left\| \begin{matrix} \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{\varphi r} \end{matrix} \right\| \sim \frac{\overset{\circ}{k}_I}{\sqrt{2\pi r}} \left\| \begin{matrix} \cos\frac{\overset{\circ}{\varphi}}{2}(1+\sin^2\frac{\varphi}{2}) \\ \cos^3\frac{\varphi}{2} \\ \sin\frac{\varphi}{2}\cos^2\frac{\varphi}{2} \end{matrix} \right\| \quad (k_I = \sigma_{22}^{\infty}\sqrt{\pi a}) \tag{6.2}$$

For a transverse-shear crack the asymptotic forms for the conventional stresses differ from (6.1) by having $\overset{\circ}{\sigma}{}_{22}^{\infty}$ replaced by $\overset{\circ}{\sigma}{}_{12}^{\infty}$, while for linear stresses we have

$$\left\| \begin{matrix} \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{\varphi r} \end{matrix} \right\| \sim \frac{k_{II}}{\sqrt{2\pi r}} \left\| \begin{matrix} \sin\frac{\varphi}{2}\left(1-3\sin^2\frac{\varphi}{2}\right) \\ -3\sin\frac{\varphi}{2}\cos^2\frac{\varphi}{2} \\ \cos\frac{\varphi}{2}\left(1-3\sin^2\frac{\varphi}{2}\right) \end{matrix} \right\| \quad (k_{II} = \sigma_{12}^{\infty}\sqrt{\pi a}) \tag{6.3}$$

It can be seen from these relations that when $\overset{\circ}{\sigma}{}_{22}^{\infty} \leqslant \sigma_{22}^{\infty}$, $\overset{\circ}{\sigma}{}_{12}^{\infty} \leqslant \sigma_{12}^{\infty}$, i.e. for medium stresses at infinity (which is usually satisfied) we can assume that the stress intensity factors are identical. Hence the conclusion follows (which is obviously of a general character) that the use of the linear crack theory is justified in problems where a single parameter—the stress intensity factor—is decisive.

Nevertheless, a difference in the dependences on the polar angle $\mathring{\varphi}(\varphi)$, even in such a symmetrical problem as a tensile crack (6.1), (6.2), can considerably influence the consideration of the finer problems of the theory. Thus, under Kelly–Tyson–Cottrell brittle-fracture conditions [2], a correction of the order of 30% is introduced when geometrical non-linearity is taken into account. A comparison of the corresponding expressions for a transverse-shear crack shows an even greater difference in the dependences on the polar angle.

7. For an arc crack, with load-free sides (Fig. 3), considered by Khristenko, and a stress at infinity of $\mathring{\sigma}^{\infty}$, directed along the normal to the crack at its tip, we have for the conventional stresses, asymptotic forms which differ from (6.1) by having $\mathring{\varphi}$ replaced by $\mathring{\varphi} + \mathring{\beta}$ and $\mathring{k}_1$ replaced by $\mathring{k}\cos(\beta/2)$. Hence, the effect of the curvature of the crack tip is taken into account by the factor $\cos(\mathring{\beta}/2)$. Hence it follows that the rectilinear crack corresponding to $\beta = 0$ is more dangerous than the corresponding curvilinear cracks.

For a plane with a cut in the form of a symmetrical lune with aperture angle $2\mathring{\beta}$ (Fig. 4), considered by Litvinekov [2], we have

$$\left\| \begin{matrix} \mathring{\sigma}_{\mathring{r}\mathring{r}} \\ \mathring{\sigma}_{\mathring{\varphi}\mathring{\varphi}} \\ \sigma_{\mathring{\varphi}\mathring{r}} \end{matrix} \right\| \sim 16k^2\sqrt{(\mathring{\sigma}_{22}^{\infty})^2 + (\mathring{\sigma}_{12}^{\infty})^2}\left(\frac{2a}{\mathring{r}}\right)^{1-k} \left\| \begin{matrix} \sin^2(1-k)\mathring{\varphi} \\ -\cos^2(1-k)\mathring{\varphi} \\ \sin(1-k)\mathring{\varphi}\cos(1-k)\mathring{\varphi} \end{matrix} \right\| \tag{7.1}$$

$$\left(k = \frac{\pi}{2(\pi - \mathring{\beta})}\right)$$

and the singularity factor is defined by the formula

$$m(\mathring{\beta}) = \frac{\frac{1}{2} - \mathring{\beta}/\pi}{1 - \mathring{\beta}/\pi} \tag{7.2}$$

This expression also holds for a plane with a wedge-shaped cut (an inclusion). This enables us to assume that the singularity factor is independent of the specific form of the neighbourhood of the corner point, and is determined solely by the value of the aperture angle.

The case of a wedge-shaped cut when one of its sides is fixed while the other is load-free is of interest. Here

$$m(\mathring{\beta}) = \frac{\frac{3}{4} - \mathring{\beta}/\pi}{1 - \mathring{\beta}/\pi} \tag{7.3}$$

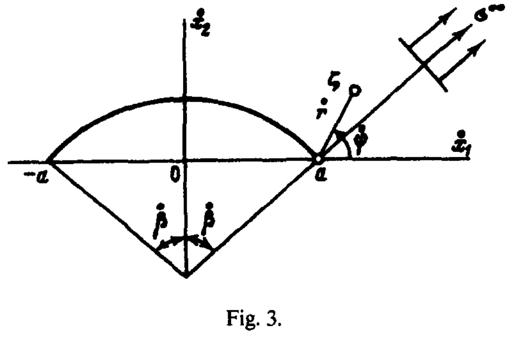

Fig. 3.

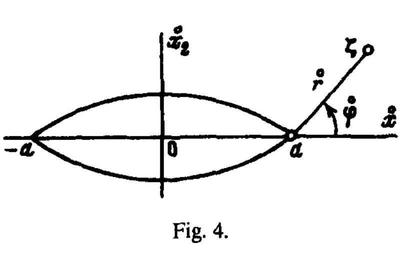

Fig. 4.

In the case of an expanded angle $m(\pi/2) = 1/2$. The point where the boundary condition is replaced by the smooth part of the boundary of the region corresponds to this.

8. A curious result was obtained when considering a longitudinal-shear crack. Here we have the following asymptotic forms.

Linear stresses

$$\left\| \begin{matrix} \sigma_{r3} \\ \sigma_{\varphi 3} \end{matrix} \right\| \sim \frac{k_{\rm III}}{\sqrt{2\pi r}} \left\| \begin{matrix} \sin\frac{\varphi}{2} \\ \cos\frac{\varphi}{2} \end{matrix} \right\| \quad (k_{\rm III} = \sigma_{23}^{\infty}\sqrt{\pi a}) \tag{8.1}$$

and conventional stresses

$$\left\| \begin{matrix} \overset{\circ}{\sigma}_{\mathring{r}\mathring{r}} \\ \overset{\circ}{\sigma}_{\mathring{\varphi}\mathring{\varphi}} \\ \overset{\circ}{\sigma}_{\mathring{\varphi}\mathring{r}} \end{matrix} \right\| \sim \frac{\overset{\circ}{k}_{\rm III}}{\sqrt{2\pi \mathring{r}}} \lambda^{1/2} \left\| \begin{matrix} \sin^2\frac{\mathring{\varphi}}{2} \\ \cos^2\frac{\mathring{\varphi}}{2} \\ \sin\frac{\mathring{\varphi}}{2}\cos\frac{\mathring{\varphi}}{2} \end{matrix} \right\| \tag{8.2}$$

$$\overset{\circ}{\sigma}_{33} \sim \frac{\overset{\circ}{k}_{\rm III}}{\sqrt{2\pi \mathring{r}}}(-\lambda^{-1}) \quad (\overset{\circ}{k}_{\rm III} = \sigma_{23}^{\infty}\sqrt{\pi a})$$

$$\left\| \begin{matrix} \overset{\circ}{\sigma}_{\mathring{r}3} \\ \overset{\circ}{\sigma}_{\mathring{\varphi}3} \end{matrix} \right\| \sim \mu(\lambda + \lambda^{-1/2}) \left\| \begin{matrix} \sin\frac{\mathring{\varphi}}{2} \\ \cos\frac{\mathring{\varphi}}{2} \end{matrix} \right\|$$

Interesting features of a longitudinal-shear crack follow from these relations. Thus, the conventional stresses $\overset{\circ}{\sigma}_{\mathring{r}\mathring{r}}$, $\overset{\circ}{\sigma}_{\mathring{\varphi}\mathring{\varphi}}$, $\overset{\circ}{\sigma}_{\mathring{\varphi}\mathring{r}}$, $\overset{\circ}{\sigma}_{33}$ possess a standard singularity and structure, similar to that for quantities in tensile and transverse-shear cracks, which have no analogy in linear theory. This indicates a way of unifying the fracture criteria for all three types of cracks. At the same time, finite conventional stresses correspond to the unique linear stresses, which have standard singularities.

9. In the non-linear approach there is a considerable difference between the follow-up load (the normal pressure σ_0) and the "dead" load (normal to the undeformed cut $\sigma^{\wedge}$).

For a normal pressure, the cut transfers [8] into the neighbourhood of the radius

$$R = \frac{R_0}{1 - \sigma_0/\alpha}, \quad R_0 = \frac{(1 + \sigma^*/\alpha)a}{2} \tag{9.1}$$

and the asymptotic form at the crack tip is

$$\left. \frac{\overset{\circ}{\sigma}_{\mathring{\varphi}\mathring{\varphi}}}{\sigma^*} \right|_{\mathring{\varphi}=0} \sim \frac{2 + (\alpha/\sigma^* - 1)(\sigma_0/\alpha)}{2(1 - \sigma_0/\alpha)} \left(\frac{a}{2\mathring{r}}\right)^{1/2} \tag{9.2}$$

For the initially normal load we have

$$R = R_0\left(1 + \frac{\alpha}{\sigma^*}\frac{\sigma^{\wedge}}{\alpha}\right), \quad R_0 = \frac{(1 + \sigma^*/\alpha)a}{2} \tag{9.3}$$

$$\left.\frac{\overset{\circ}{\sigma}_{\overset{\circ}{\phi}\overset{\circ}{\phi}}}{\sigma^*}\right|_{\overset{\circ}{\phi}=0} \sim \left(1+\frac{\alpha}{\sigma^*}\frac{\sigma^\wedge}{\alpha}\right)\left(\frac{a}{2\overset{\circ}{r}}\right)^{1/2} \tag{9.4}$$

Hence, quite different results are obtained with the loads considered.

The asymptotic form (9.4) holds for the linear approach. We have a completely unacceptable result for the shape of the deformation crack: a horizontal cut changes into a vertical one with overlapping sides.

10. The greatest contrast between the solutions using the linear and non-linear theories occurs for mixed types of cracks. Thus, for the rectilinear crack considered above we have for the conventional stresses, asymptotic forms which differ from (6.1) in that k_I° is replaced by $\sqrt{((k_I^\circ)^2 + (k_{II}^\circ)^2)}$.

According to the criterion for crack growth in the direction of maximum tensile stresses (the Erdogan–Sih criterion) it follows from the expression for $\overset{\circ}{\sigma}_{\overset{\circ}{\phi}\overset{\circ}{\phi}}$ that, irrespective of the ratio k_{II}°/k_I°, a crack should start to grow "along itself", i.e. along a material fibre $\overset{\circ}{\phi} = 0$, which continues the cut. This, at first sight, strange result is in clear contradiction with the discontinuity of a mixed type of crack, "observed" in the confirmation of the linear theory.

In the geometrically non-linear approach, this apparent paradox receives a natural explanation [1, 2]. A crack, in fact, starts to grow along a material fibre which continues the cut. Simultaneously, however, a considerable rotation of the neighbourhood of the crack tip occurs. Just the interaction between these two effects creates the semblance of a discontinuity. Note that this applies to brittle fracture. In experiments carried out on rubber plates no discontinuity is observed after the load is removed. The crack starts to grow in the direction of its continuation and then gradually bends. Note that the thumb rule [2], which is considered to be roughly approximate in the linear theory, is satisfied exactly in the non-linear approach.

Further, according to the expression for $\overset{\circ}{\sigma}_{\overset{\circ}{\phi}\overset{\circ}{\phi}}$, the non-linear criterion for crack growth in the direction of maximum tensile stresses is formulated as follows:

$$\sqrt{(k_I^\circ)^2 + (k_{II}^\circ)^2} = k_{IC}^\circ \quad \text{for } \overset{\circ}{\phi} = 0$$

which is identical with Irwin's criterion.

11. In the linear approach, as it applies to a tensile crack, the following discrete fracture criterion (Fig. 2) was formulated [9, 10]

$$\frac{1}{D}\int_0^D \sigma_{22}(\xi, 0), d\xi \leqslant \sigma_c \quad (\xi = x_1 - a) \tag{11.1}$$

where D is the atomic diameter and σ_c is the ideal strength of the material. In fact, a (moderately) large deformation was considered in [9, 10]. Moreover, in the fracture of the atomic lattice, its transverse strain was ignored, as was pointed out in [11]. Hence, strictly speaking, criterion (11.1) must be changed, assuming

$$\frac{1}{D}\int_0^D \overset{\circ}{\sigma}_{22}(\xi, 0), d\xi \leqslant \sigma_c \quad (\xi = \overset{\circ}{x}_1 - a) \tag{11.2}$$

This criterion can be extended in a natural way to the case of an arbitrary crack (a cut)

$$\frac{1}{D}\int_0^D \overset{\circ}{\sigma}_{\overset{\circ}{\phi}_{max}\overset{\circ}{\phi}}(\xi, 0) d\xi \leqslant \sigma_c \tag{11.3}$$

Here we initially find the direction ($\overset{\circ}{\phi}$) in which the quantity $\sqrt{(2\pi r^\circ)}\, \overset{\circ}{\sigma}_{\overset{\circ}{\phi}\overset{\circ}{\phi}}$ has its maximum value, while ξ is the distance along this direction. In the general case D is a characteristic structural dimension of the material (a composite) considered.

12. To take into account the general (geometrical and physical) non-linearity, an nth order slightly compressed material has been proposed [1, 2], for which

$$\Phi = \frac{En^{-2}}{1+\nu}\left[-(\mathrm{III}_{\Lambda^n}-1)+\Psi(\mathrm{I}_{\Lambda^n})\right]$$

$$(I_{\Lambda^n} = \lambda_1^n+\lambda_2^n+\lambda_3^n,\ \mathrm{III}_{\Lambda^n} = \lambda_1^n\lambda_2^n\lambda_3^n = J^n) \tag{12.1}$$

Here λ_i are the principal elongation multiplicities, $\Psi(I_{\Lambda^n})$ is an arbitrary real function, "responsible" for the deformation, while the invariant III_{Λ^n} takes into account in the simplest (linear) way the small change in volume, J is the multiplicity of the change in volume and n is a constant characterizing the non-linearity of the material.

The conditions

$$\Psi'(3)=1, \quad \Psi''(3)=\frac{1-\nu}{1-2\nu} \tag{12.2}$$

ensure that for small strains, the law of elasticity changes into Hooke's law. In the plane problem, with $n = 1$, the equilibrium equation is reduced to quadratures

$$\frac{\partial z}{\partial \zeta} = \frac{\chi[\xi(\zeta,\lambda)\overline{\xi(\zeta,\lambda)};\lambda]\ \xi(\zeta,\lambda)}{\overline{\xi(\zeta,\lambda)}}$$

$$\frac{\partial z}{\partial \bar\zeta} = \int\frac{\partial}{\partial \bar\zeta}\left[\frac{\chi[\xi(\zeta,\lambda)\overline{\xi(\zeta,\lambda)};\lambda]}{\overline{\xi(\zeta,\lambda)}}\right]\xi(\zeta,\lambda)d\zeta + \eta'(\bar\zeta,\lambda) \tag{12.3}$$

$$z = \frac{\int\chi[\xi(\zeta,\lambda)\ \overline{\xi(\zeta,\lambda)};\lambda]\ \xi(\zeta,\lambda)d\zeta}{\overline{\xi(\zeta,\lambda)}} + \eta(\bar\zeta,\lambda)$$

Here $\xi(\zeta, \lambda)$, $\eta(\zeta, \lambda)$ and the real $\chi[\cdot]$ are arbitrary functions. A reasonable choice of these gives families of exact solutions of the (geometrically and physically) non-linear plane problem.

13. The results obtained when setting up the non-linear crack theory was used in a natural way for a non-linear consideration of rectilinear edge dislocations, wedge-shaped disclinations and point forces. Anisotropic materials were considered (mainly orthotropic and transversally isotropic materials, including crystals of the cubic and hexagonal systems). We also investigated numerous defects of the actual stresses and their absence in conventional stresses. The results obtained differ considerably (often qualitatively) from their linear analogues.

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